Theoretical frameworks for the SIRS model in complex networks with different localization patterns

José Carlos M. Silva,[1] Diogo H. Silva,[2] Francisco A. Rodrigues,[2] and Silvio C. Ferreira[1,3]

[1] *Departamento de Física, Universidade Federal de Viçosa, 36570-900 Viçosa, Minas Gerais, Brazil*

[2] *Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos, SP 13566-590, Brazil*

[3] *National Institute of Science and Technology for Complex Systems, 22290-180, Rio de Janeiro, Brazil*

In the present work, we investigate the performance of theoretical approaches in the prediction of properties of the SIRS epidemic model, which involves immunity periods $1/\alpha$ and whose acronym indicates the allowed states (*susceptible*, *infected* and *recovered*) as well as the transitions among them.

As a special case, we have the SIS model ($1/\alpha = 0$), whose activation mechanisms in power-law degree distributed networks, $P(k) \sim k^{-\gamma}$, are well known. They involve self-sustained activity by means of feedback mechanisms in subextensive subgraphs [1], thus spreading to the rest of the network, with the epidemic threshold vanishing in the thermodynamic limit [2, 3]. The *quenched mean-field theory* [4] predicts this behavior qualitatively, with the threshold $\lambda_c = \frac{1}{\Lambda_A}$ vanishing in the thermodynamic limit, where Λ_A is the Largest Eigenvalue (LEV) of the adjacency matrix A_{ij}. An enhancement of the QMF theory is the *pair-quenched mean-field theory* (PQMF), which takes into account the dynamic correlations at a pairwise level [5].

In the SIR limit ($1/\alpha = \infty$), there is a transition to a state in which the fraction of recovered agents is finite, with a vanishing epidemic threshold in power-law distributed networks with $\gamma \leq 3$ and a finite one when $\gamma > 3$. This behavior is well captured by the *heterogeneous mean-field theory* and the exact threshold is predicted by the *message passing* approach [6] on top of tree-like networks.

It is not clear which theoretical approach would be more suitable for the SIRS model with finite immunity periods $1/\alpha$, what is the main concern of our present work [7]. To investigate their properties, we developed PQMF equations for the SIRS model on networks and the rDMP equations reported in [8], whose predictions for threshold and Inverse Participation Ratio (IPR) [9, 10] were compared with extensive stochastic simulations. For more details, we refer the reader to [7].

Our results show that PQMF theory outperforms other approaches in networks without considerable localization effects [7]. Fig. 1 shows how localization affects mean-field theory predictions, by introducing a *hub* in a *random regular* network. PQMF theory undergoes strong localization on the hub and its neighborhood, while rDMP theory shows less localized behavior, agreeing qualitatively with simulation results, specially in dynamics with longer immunity times $1/\alpha$. Other results shown

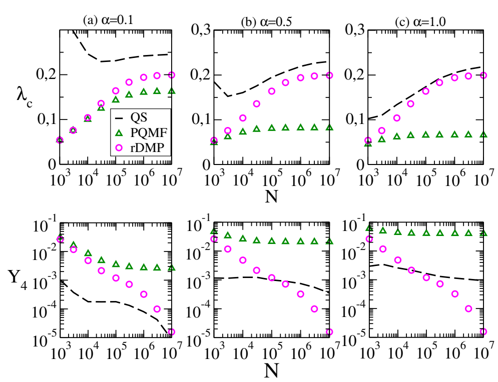

FIG. 1. Finite size scaling for SIRS model in random regular networks ($k_{RR} = 6$) with hub ($k_{hub} = 10^3$). Upper row: threshold λ_c. Lower row: IPR at the threshold. Immunity rates α indicated at the column heading.

in our paper [7] present the same tendency of localization for the PQMF theory in the asymptotic limit, predicting a vanishing epidemic threshold, while simulations and rDMP theory predict delocalization and a finite threshold in power-law degree distributed networks with $\gamma > 3$ [7].

Keywords: Epidemiology - Mathematical models. Networks (Mathematics). SIRS model. Immunity period. Mean-field theory.

ACKNOWLEDGMENTS

The authors thank the financial support given by: *Coordenação de Aperfeiçoamento de Pessoal de Nível Superior* (CAPES), *Conselho Nacional de Desenvolvimento Científico e Tecnológico* (CNPq), *Fundação de Amparo à Pesquisa do Estado de São Paulo* (FAPESP) and *Fundação de Amparo à Pesquisa do Estado de Minas Gerais* (FAPEMIG) - Brazil.

[1] C. Castellano and R. Pastor-Satorras, Competing activation mechanisms in epidemics on networks, Scientific Reports **2**, 371 (2012).

[2] M. Boguñá, C. Castellano, and R. Pastor-Satorras, Nature of the epidemic threshold for the susceptible-infected-susceptible dynamics in networks, Phys. Rev. Lett. **111**, 068701 (2013).

[3] S. C. Ferreira, R. S. Sander, and R. Pastor-Satorras, Collective versus hub activation of epidemic phases on networks, Phys. Rev. E **93**, 032314 (2016).

[4] F. Chung, L. Lu, and V. Vu, Spectra of random graphs with given expected degrees, PNAS (2003).

[5] A. S. Mata and S. C. Ferreira, Pair quenched mean-field theory for the susceptible-infected-susceptible model on complex networks, Europhysics Letters **103**, 48003 (2013).

[6] B. Karrer and M. E. Newman, Message passing approach for general epidemic models, Physical Review E (2010).

[7] J. C. M. Silva, D. H. Silva, F. A. Rodrigues, and S. C. Ferreira, Comparison of theoretical approaches for epidemic processes with waning immunity in complex networks, Phys. Rev. E **106**, 034317 (2022).

[8] M. Shrestha, S. V. Scarpino, and C. Moore, Message-passing approach for recurrent-state epidemic models on networks, Physical Review E **92**, 022821 (2015).

[9] A. V. Goltsev, S. N. Dorogovtsev, J. G. Oliveira, and J. F. F. Mendes, Localization and spreading of diseases in complex networks, Phys. Rev. Lett. **109**, 128702 (2012).

[10] D. H. Silva and S. C. Ferreira, Dissecting localization phenomena of dynamical processes on networks, Journal of Physics: Complexity **2**, 025011 (2021).